A ring would mean a commutative ring with identity unless specified otherwise.

1. (15 points) State true or false.
   (a) Every nonzero ring has a prime ideal.
   (b) Every Euclidean domain is a Principal Ideal Domain.
   (c) Every Unique Factorization Domain is a Principal Ideal Domain.
   (d) A prime element in an integral domain is irreducible.
   (e) Two distinct nonzero prime ideals are co-maximal ideals.

   Solution:
   (a) T
   (b) T
   (c) F
   (d) T
   (e) F

2. Let $\mathbf{R}$ be an integral domain. Which of the following are an integral domain? No justification needed.
   (a) $\mathbf{R} \oplus \mathbf{R}$
   (b) $\mathbf{R}[X]$, the polynomial ring over $\mathbf{R}$.
   (c) $S^{-1}\mathbf{R}$ where S is a multiplicative subset of $\mathbf{R}$ not containing 0.
   (d) Any ring A such that $\mathbf{R}$ is a subring of A.

   Solution: $\mathbf{R}[X]$ and $S^{-1}\mathbf{R}$ are the only integral domains $\square$

3. Are the two rings $\mathbf{Q}[X]/(X^2-1)$ and $\mathbf{Q}[\mathbf{X}]/(X^2+1)$ isomorphic?

   Solution: Let $I = (X^2 - 1)$ Notice that $(X - 1 + I).(X + 1 + I) = X^2 - 1 + I$. It follows that $Q[X]/I$ is not an integral domain. On the other side $X^2 + 1$ is prime in $Q[X]$, thus $Q[X]/(X^2 + 1)$ is an integral domain. Hence two rings $Q[X]/(X^2-1)$ and $Q[X]/(X^2+1)$ can not be isomorphic. $\square$

4. (4+16=20 points) Define prime ideals and maximal ideals. Show that in a ring with finitely many elements every prime ideal is maximal ideal.

   Solution: Let R be a commutative ring unit unit 1. Then an ideal $I \neq \{0\}$ in R is called *prime ideal* if $ab \in I$ implies either $a \in P$ or $b \in P$. An ideal I in R is called *maximal ideal* if there does any any idea J such that $I \subsetneq J \subsetneq R$.

Let I be a prime ideal in ring R, where R has only finitely many element. Then R/I is an integral domain which contains only finitely many element. It assures that R/I is filed. Hence I is a maximal ideal in R. $\square$

5. Let k be a field and $R = k[x]$ be the polynomial ring. Let $S = \{1; x; x^2; \cdots\}$ be a multiplicative subset. Show that the rings $R[y]/(xy-1)$ and $S^{-1}R$ are isomorphic rings.

Solution: We define a map $\varphi : R[y]/(xy-1) \to S^{-1}R$ given by

$$\varphi(p(x,y)) = p(x, \frac{1}{x}) \ \ \forall p(x,y) \in R[y]. \tag{1}$$

It is straight forward to see that the map φ is a linear algebra homeomorphisom. By the first isomorphisom theorem, there exists $R[x]/\ker(\varphi)$ is isomorphic to $S^{-1}R$. Moreover, since $\varphi(xy-1) = 0$, thus the ideal $(xy-1) \subseteq \ker(\varphi)$. Therefore, it is sufficient to show that

$$(xy-1) \supseteq \ker(\varphi).$$

In order tho show that, let $p(x,y) \in R[y]$ such that $p(x, \frac{1}{x}) = 0$. Now, one can write $p(x,y)$ as

$$p(x,y) = (xy)^n q_1(x) + (xy)^m q_2(y), \tag{2}$$

where $q_1(x) \in K[x], q_2(y) \in K[y], \ and\ n, m \in \mathbb{N} \cup \{0\}$. Then we can write from Equation (2) that

$$p(x,y) + I = q_1(x) + q_2(y) + I.$$

This implies that

$$p(x,y) = q_1(x) + q_2(y) + h(xy-1),$$

where $h(0) = 0$. Since $p(x, \frac{1}{x}) = 0$, we have

$$q_1(x) + q_2(\frac{1}{x}) = 0. \tag{3}$$

By applying fundamental calculus sincerely, we have $q_1(x) = 0 = q_2(x)$. Hence we have

$$p(x,y) = h(xy-1) \in \ker(\varphi).$$

This completes the proof. $\square$